Nonlinear Heat Conductions in Rarefied Gases

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Theme

NONLINEAR problems of thermal conduction in monatomic rarefied gas confined within two concentric cylinders and spheres are studied by the four-moment method coupled with the bimodal two-stream distribution function of Beck. Finite temperature difference between the two boundary surfaces is included in the present theory. Therefore, present results are presumed to be complete solutions obtainable by the four-moment technique with the assumed velocity distribution of the gas.

Content

Recently, Lees' four-moment approach[1] for heat conduction problem in rarefied gases was further extended by Lou and Shih[2] to investigate nonlinear problems. Instead of linearizing the problem, they employed the technique of series expansion. It was shown that the nonlinear solutions could be obtained in analytic forms. In principle, their method is suitable for any temperature difference. However, as the temperature difference is getting larger, the solutions will have to include more higher-order terms in the series; hence, they become cumbersome. In order to overcome this inconvenience, another modified four-moment approach is proposed.

Let us consider the steady heat conduction problem in a rarefied Maxwellian gas confined between two concentric cylinders, or spheres. The surface of the inner cylinder (or sphere) with radius R_I is maintained at a uniform constant temperature T_I; and the inner surface of the outer cylinder (or sphere) with radius R_{II} is maintained at T_{II}. The velocity distribution function of gas molecules $f(V, R)$ is assumed to be the two-steam bimodal distribution function.[3]

In region 1

$$f_1 = \gamma_{1a} f_I + \gamma_{1b} f_{II} \tag{1}$$

and in region 2

$$f_2 = \gamma_{2a} f_I + \gamma_{2b} f_{II} \tag{2}$$

where

$$f_{I,II} = n_{I,II}(m/2\pi k T_{I,II})^{3/2} \exp(-mV^2/2kT_{I,II})$$

and $\gamma_{1a}, \gamma_{1b}, \gamma_{2a}, \gamma_{2a}, \gamma_{2b}$, are the four unknown mixing functions to be determined. By using Eqs. (1) and (2), and considering the four basic moments, one can easily get the following equations from Maxwell's transport equation.[4]

Continuity

$$(\gamma_{1a} - \gamma_{2a}) + (\eta_{1b} - \eta_{2b})t = 0 \tag{3}$$

Radial-momentum

$$(d/d\bar{R})[(\gamma_{1a} + \gamma_{2a}) + (\eta_{1b} + \eta_{2b})t^2] + F_j(\alpha)(d/d\bar{R})[(\gamma_{1a} - \gamma_{2a}) + (\eta_{1b} - \eta_{2b})t^2] = 0 \tag{4}$$

Energy

$$(\gamma_{1a} - \gamma_{2a}) + (\eta_{1b} - \eta_{2b})t^3 = \delta_j(1 - t^2) \tag{5}$$

Radial-heat-flux

$$(d/d\bar{R})[(\gamma_{1a} + \gamma_{2a}) + (\eta_{1b} + \eta_{2b})t^4] + F_j(\alpha)(d/d\bar{R})[(\gamma_{1a} - \gamma_{2a}) + (\eta_{1b} - \eta_{2b})t^4] + (K\delta_j/\bar{R}^j)\{[\gamma_{1a} + \eta_{1b}][1 - G_j(\alpha)] + [\gamma_{2a} + \eta_{2b}][1 + G_j(\alpha)]\} = 0 \tag{6}$$

where

$$t = (T_{II}/T_I)^{\frac{1}{2}}; \quad \bar{R} = R/R_I; \quad K = \tfrac{4}{15}(R_I/\lambda_I) = \tfrac{2}{15}Kn^{-1}$$
$$\eta_{1b} = \gamma_{1b}(n_{II}/n_I); \quad n_{2b} = \gamma_{2b}(n_{II}/n_I)$$

and $j = 1$ for the cylindrical problem, $j = 2$ for the spherical problem, with

$$G_1(\alpha) = 2\alpha/\pi; \quad G_2(\alpha) = \sin\alpha; \quad F_1(\alpha) = (1/\pi)(\sin 2\alpha - 2\alpha)$$
$$F_2(\alpha) = -\sin^3\alpha; \quad \alpha = \cos^{-1}(R_I/R)$$

The parameter δ_j is an integration constant proportional to the radial heat flux. Assuming an incomplete accommodation a at the inner surface, one can solve for $\gamma_{1a}, \gamma_{2a}, \gamma_{1b}$, and γ_{2b} and obtain

$$\begin{aligned}
\gamma_{1a} &= [1 - I_j(\bar{R})]e^{K_b\delta_j H_j(\bar{R})} + [K_0/K_b + (1/a - \tfrac{1}{2})K_a\delta_j/K_b][1 - e^{K_b\delta_j H_j(\bar{R})}] - \delta_j(1/a - 1)\\
\gamma_{2a} &= \gamma_{1a} - \delta_j\\
\gamma_{1b} &= (n_I/t^2 n_{II})[1 - \gamma_{1a} - (1/a - 1)(1 - t)\delta_j]\\
\gamma_{2b} &= (n_I/t^2 n_{II})[1 - \gamma_{1a} - (1/a - 1)(1 - t)\delta_j + t\delta_j]
\end{aligned} \tag{7}$$

where

$$K_0 = Kt^{-2}, \quad K_a = K(t^{-1} - 1), \quad K_b = K(t^{-2} - 1);$$
$$H_1(\bar{R}) = \ln\bar{R}; \quad H_2(\bar{R}) = (1 - 1/\bar{R})$$
$$I_j(\bar{R}) = \frac{K_a\delta_j^2}{2}\int_1^R e^{-K_b\delta_j H_j(\bar{R})}G_j(\alpha)\frac{d\bar{R}}{\bar{R}^j}$$

The radial heat flux constant δ_j should satisfy the following equation:

$$\delta_j = a[1 - I_j(\bar{R}_{II})]e^{K_b\delta_j H_j(\bar{R}_{II})} + a\left[\frac{K_0}{K_b} + \left(\frac{1}{a} - \frac{1}{2}\right)\frac{K_a\delta_j}{K_b}\right][1 - e^{K_b\delta_j H_j(\bar{R}_{II})}] \tag{8}$$

From Ref. 4, the mean temperature $\bar{T}(R)$ and the radial heat flux $q_R(R)$ can be derived as

$$q_R/q_{Rfm} = \delta_j \tag{9}$$

and

$$\frac{T - T_{II}}{T_I - T_{II}} = \frac{[\gamma_{1a} - \frac{1}{2}\delta_j(1 + G_j(\alpha))]t^2}{1 - \gamma_{1a}(1 - t^2) - (1/a - 1)(1 - t)\delta_j + \frac{1}{2}t(1 - t)(1 + G_j(\alpha))\delta_j} \tag{10}$$

The linearized solutions[1,4] are recovered by taking $t = 1$.

It is clear, from Eqs. (8–10) that if one finds δ_j for each given set of values of Kn, t and $\bar{R}_{II}$, one will be able to calculate the heat-transfer ratio and the temperature distribution in the gas. The correct value of δ_j can be obtained from Eq. (8) by a simple iteration procedure. One can use the linearized solution[1,4] or

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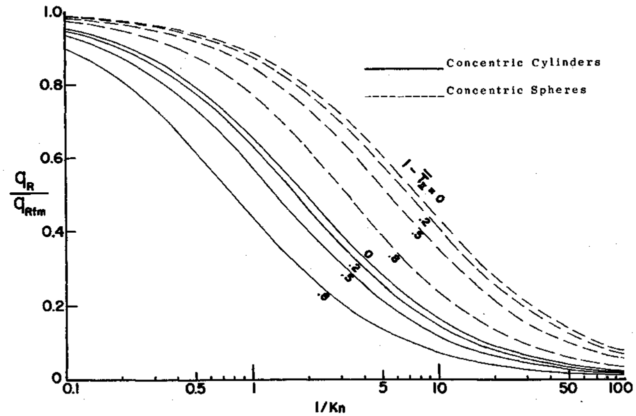

Fig. 1 Results of q_R/q_{Rf_m} vs Kn^{-1} with $R_{II} = 50$ and $a = 1$.

better yet the second-order series solution[2] as the first trial value for the iteration process. The calculations involved in the present work are carried out to an accuracy of 10^{-6}.

The effect of finite temperature difference on the heat conduction ratio is shown in Fig. 1. The heat conduction ratio q_R/q_{Rfm} is plotted against the inverse Knudsen number Kn^{-1} for several values of the temperature difference. In both cases of concentric cylinders and spheres, the ratio q_R/q_{Rfm} is found to be decreasing as the temperature difference increases. The net head conduction rate is however higher for the case of a larger temperature difference since q_{Rfm} is greater. It should be noted that the results for large temperature differences may only be useful for cases of small values of T_I. This is because that a large temperature difference will otherwise result in large temperature gradient in the gas, and the transfer mechanism other than pure conduction will be introduced.

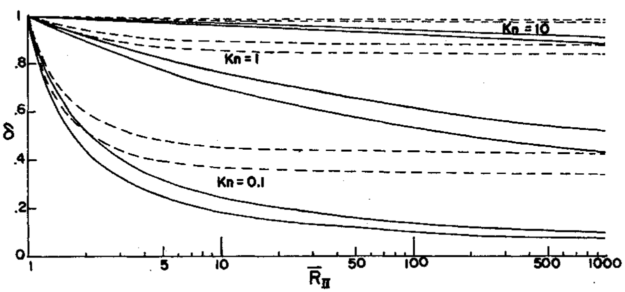

Fig. 2 Results of q_R/q_{Rf_m} vs R_{II} for several values of Kn; solid lines for cylinders and dashed line for spheres; upper curves for linear and lower curves for nonlinear case of $1 - T_{II} = 0.5$.

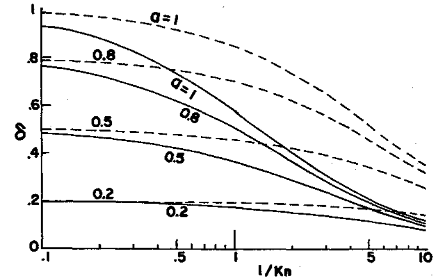

Fig. 3 Effect of thermal accommodation on heat conduction with $R_{II} = 50$ and $1 - T_{II} = 0.5$.

The effect of configuration on the heat conduction is shown in Fig. 2. It is interesting to observe that the dependence of q_R/q_{Rfm} and R_{II} is almost a straight line for both geometries when the Knudsen number is large. As one departs from this region of large Kn, this dependence is no longer linear. It implies, therefore, the existence of a more complex relationship between the q_R/q_{Rfm} and $\bar{R}_{II}$.

In Fig. 3 the heat conduction ratio q_R/q_{Rfm} is shown for several values of the thermal accommodation coefficient at the inner solid surfaces with a temperature difference of 0.5, and $\bar{R}_{II} = 50$. The effect of incomplete thermal accomodation is shown to be important for gases in the near-free molecular limit, i.e., $Kn \sim 1$. As the gas approaches to the continuum limit of small Knudsen number, the incomplete thermal accomodation ceases to be a dominant factor on the heat conduction rate, especially for $a > 0.8$. This is because the increasing influence on the heat conduction rate of the ever increasing number of collisions a gas molecule must encounter as it moves toward the outer solid boundary.

References

[1] Lees, L., "Kinetic Theory Description of Rarefied Gas Flow," *Journal of Society of Industrial and Applied Mathematics,* Vol. 13, No. 1, 1965, pp. 278–311.

[2] Lou, Y. S. and Shih, T. K., "Nonlinear Heat Conduction Problems in Rarefied Gases Confined Between Concentric Cylinders and Spheres," *The Physics of Fluid,* May 1972, pp. 785–788.

[3] Beck, J. W., "Bimodal Two-Stream Distribution and Compressible Couette Flow," Fourth International Symposium on Rarefied Gas Dynamics, edited by J. H. deLeeuw, Vol. 1, 1965, pp. 354–369.

[4] Lou, Y. S. and Shih, T. K., "On Model Solutions of Linearized Heat Conduction Problems In Rarefied Gases," ASME Paper 69-WA/HT-17, Los Angeles, Calif., Nov. 1970.